Convex embeddability on linear/circular orders and connections to knot theory

(joint work with Vadim Kulikov, Alberto Marcone and Luca Motto Ros)

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- We say that E and F are Borel bi-reducible, and write $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$.

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- If X and Y are two topological spaces, we say that E is **Baire reducible** to F, and write $E \leq_{Baire} F$, if there exists a Baire measurable map $\varphi : X \to Y$ reducing E to F.

Let X be a Polish or a standard Borel space. A subset $A \subseteq X$ is **analytic** (or $\mathbf{\Sigma}^1_1$) if there is a Borel subset C of $X \times \omega^\omega$ such that for all $x \in X$,

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Example

Let LO be the Polish space of codes for linear orders on ω, i.e.

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- $\cong_{LO}$ is S_∞-complete, i.e. any other equivalence relation arising from a Borel action of the group S_∞ Borel reduces to $\cong_{LO}$.

Connections between linear orders and knots

Definition

Let $\bar{B}$ be a space homeomorphic to a closed ball in $\mathbb{R}^3$. Given a map $f\colon [0;1] \to \bar{B}$, we say that the pair $(\bar{B}, \operatorname{Im} f)$ is a **proper arc** in $\bar{B}$ if f is a topological embedding and $f(x) \in \partial \bar{B} \iff x = 0$ or $x = 1$. The collection of proper arcs is denoted by $\mathcal{A}$.

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Two proper arcs $(\bar{B}, f)$ and $(\bar{B}', f')$ are **equivalent**, in symbols

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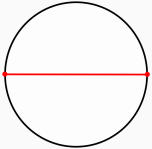
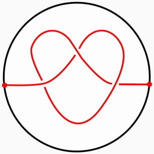


Trivial arc



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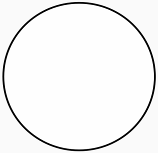

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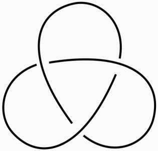

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Theorem (V. Kulikov, 2017)

(a) $\cong_{LO} \leq_B \equiv_{\mathcal{A}}, \equiv_{\mathcal{K}}$.

(b) There is a turbulent equivalence relation E such that $E \leq_B \equiv_{\mathcal{A}}, \equiv_{\mathcal{K}}$, hence $\equiv_{\mathcal{A}}, \equiv_{\mathcal{K}} \not\leq_B \cong_{LO}$. Thus $\cong_{LO} <_B \equiv_{\mathcal{A}}, \equiv_{\mathcal{K}}$.

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Let $(\bar{B}, f), (\bar{B}', g) \in \mathcal{A}$. We say that $(\bar{B}', g)$ is a **component** of $(\bar{B}, f)$, and set

$$(\bar{B}', g) \precsim_{\mathcal{A}} (\bar{B}, f),$$

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What is the counterpart of $\precsim_{\mathcal{A}}$ for linear orders?

Convex embeddability on LO

Consider the relation of **convex embeddability** $\trianglelefteq_{LO}$ between two linear orders L and L' (R. Bonnet, E. Corominas and M. Pouzet, 1973):

$$L \trianglelefteq L' \text{ if } L \text{ is isomorphic to a convex subset } \widetilde{L} \text{ of } L'.$$

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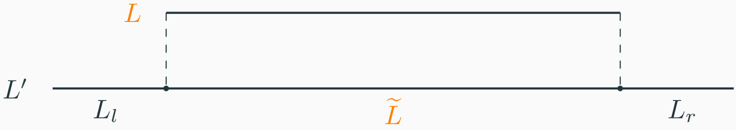

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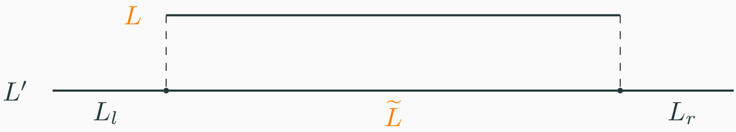

Clearly, $L \trianglelefteq_{\mathrm{LO}} L' \Rightarrow L \sqsubseteq_{LO} L'$, where $\sqsubseteq_{LO}$ is the usual embeddability on LO.

We call **convex bi-embeddability**, and denote by $\bowtie_{LO}$, the equivalence relation on LO induced by $\trianglelefteq_{LO}$.

Clearly, for $L, L' \in LO$,

$$L \cong_{LO} L' \Rightarrow L \bowtie_{LO} L',$$

but the converse is not true.

Example

$\omega + \mathbb{Z}\omega \bowtie_{\mathrm{LO}} \mathbb{Z}\omega$, but $\omega + \mathbb{Z}\omega \ncong_{LO} \mathbb{Z}\omega$.

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Theorem [I. - Motto Ros]

$\trianglelefteq_{LO} \leq_B \precsim_{\mathcal{A}}$, thus also $\bowtie_{LO} \leq_B \approx_{\mathcal{A}}$.

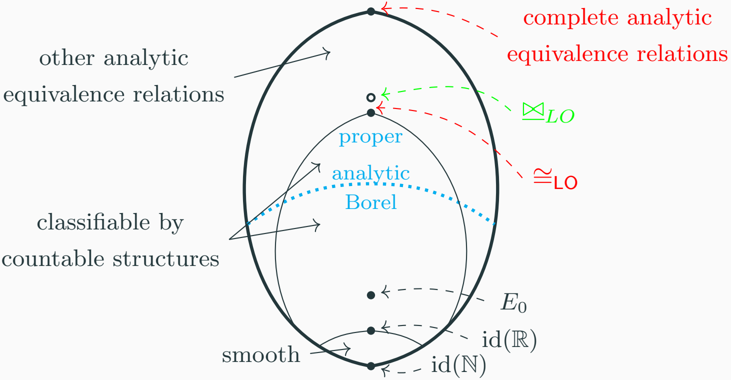
Complexity with respect to Borel Reducibility

Theorem [I. - Kulikov - Marcone - Motto Ros]

(a) $\cong_{LO} \leq_B \bowtie_{LO}$.

(b) $\bowtie_{\mathrm{LO}} \leq_{Baire} \cong_{LO}$.

(c) If X is a turbulent Polish G-space, then the equivalence relation induced by the group G on X is not Borel reducible to $\bowtie_{LO}$.

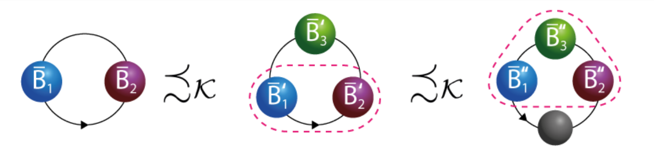


A component notion for Knots?

One may be tempted to transfer the component relation from proper arcs to knots through the transformation $(\bar{B}, f) \mapsto K_{(\bar{B},f)}$.

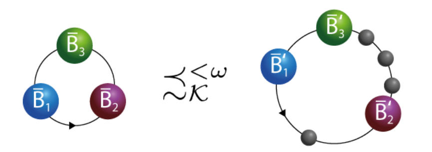
> *Given two knots $K, K' \in \mathcal{K}$, we say that K is a* ***component*** *of K', and write $K \precsim_{\mathcal{K}} K'$, if $K \equiv_{\mathcal{K}} K_{(\bar{B}',K'\cap\bar{B}')}$ for some sub-arc $(\bar{B}', K' \cap \bar{B}')$ of K'.*

However, the choice of the cutting point for a knot is in general not unique and one may produce unexpected situations. Moreover, $\precsim_{\mathcal{K}}$ is **not transitive**.



Then we consider the "transitivization" of the $\precsim_{\mathcal{K}}$ and introduce the (finite) piecewise component relation $\precsim_{\mathcal{K}}^{<\omega}$.

The (finite) piecewise mutual component relation



Definition

Let $K, K' \in \mathcal{K}$. Then K is a **(finite) piecewise component** of K', in symbols

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if and only if there is an orientation of K' and a finite number of closed balls $\bar{B}_1', ..., \bar{B}_n'$ such that

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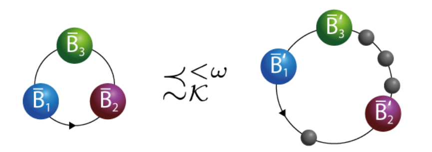

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(a) the $(\bar{B}'_i, K' \cap \bar{B}'_i)$ are (almost) pairwise disjoint sub-arcs of K', oriented according to the chosen orientation of K', of which K is an "ordered" **(finite) tame sum**;

(b) if an endpoint of some $(\bar{B}'_i, K' \cap \bar{B}'_i)$ is singular, then it is not isolated.

Countable Circular Orders

Definition (Cěch, 1969)

A ternary relation $C \subset X^3$ on a set X is said to be a **circular order** if the following conditions are satisfied:

- Cyclicity: $(x, y, z) \in C \Rightarrow (y, z, x) \in C$;
- Asymmetry: $(x, y, z) \in C \Rightarrow (y, x, z) \notin C$;
- Transitivity: $(x, y, z), (x, z, w) \in C \Rightarrow (x, y, w) \in C$;
- Totality: if $x, y, z \in X$ are distinct, then $(x, y, z) \in C$ or $(x, z, y) \in C$.

Denote by CO the Polish space of codes for circular orders on ω, i.e.

$$CO = \{C \in 2^{\omega\times\omega\times\omega} : C \text{ codes a circular order}\}.$$

The isomorphism relation on CO

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Let $C, C' \in CO$. We say that C and C' are **circularly isomorphic**, and write $C \cong_{CO} C'$, if there exists a bijective function between them which preserves the circular order.

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Every $L \in LO$ defines a standard circular order $C_L \in CO$ as follows:

$C_L(n, m, k) \iff (n <_L m <_L k) \vee (m <_L k <_L n) \vee (k <_L n <_L m)$.

Clearly, for $L, L' \in LO$,

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$\omega + 1 \ncong_{LO} \omega$, but $C_{\omega+1} \cong_{CO} C_\omega$.

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Theorem [I. - Marcone]

- $\cong_{CO} \sim_B \cong_{LO}$.
- $\cong_{CO} \leq_B \equiv_{\mathcal{K}}$.

Convex embeddability on CO

Definition (B. Kulpeshov, H. D. Macpherson, 2005)

Let $A \subseteq C$, where C is a circular order. The set A is said to be **convex** in C if for any $x, y \in A$ one of the following holds:

1. for any $z \in C$ with $C(x, z, y)$ we have $z \in A$;
2. for any $z \in C$ with $C(y, z, x)$ we have $z \in A$.

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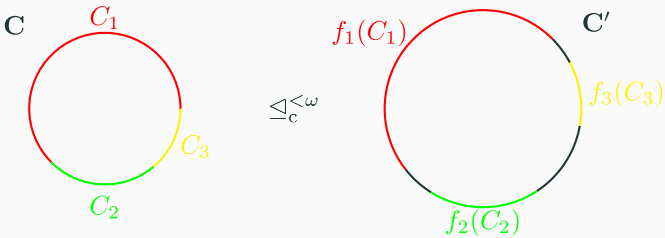
Definition

Let C and C' be circular orders. We say that C is a **convex** of C', and write $C \trianglelefteq_c C'$, if there exists a convex subset A of C' such that $C \cong_{CO} A$. We denote by $(\trianglelefteq_c)_{CO}$ the restriction of the convexity relation to the set CO of (codes for) countable circular linear orders.

The convexity relation $\trianglelefteq_{\mathrm{c}}$ on CO is **not** transitive.



Definition

Let $C, C' \in CO$. Then $C \trianglelefteq_{\mathrm{c}}^{<\omega} C'$ if and only if there exists $k \in \omega$ and (non necessarily infinite) convex subsets $C_1, \ldots, C_k$ of C such that

- $C = C_1 + ... + C_k$, and
- for every $i = 1, ..., k$ there exists $f_i : C_i \to C'$ witnessing $C_i \trianglelefteq_{\mathrm{c}} C'$ such that the $f_i(C_i)$'s are pairwise disjoint in C' and

  $$C'(f_i(x_i), f_j(y_j), f_h(z_h))$$

  for every $x_i \in C_i, y_j \in C_j, z_h \in C_h$ and $i < j < h \leq k$.

$(\trianglelefteq_{\mathrm{c}}^{<\omega})_{CO}$ is an analytic quasi-order on CO. Denote by $(\underline{\bowtie}_c^{<\omega})_{CO}$ its induced (analytic) equivalence relation.

Theorem [I. - Marcone - Motto Ros]

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$\cong_{LO} \leq_B \underline{\bowtie}_c^{<\omega}$.

Consider the equivalence relation E_1, that is defined on $2^{\omega\times\omega}$ as

$$x \; E_1 \; y \iff \exists m \; \forall n \geq m \; \forall k \; x(n,k) = y(n,k).$$

E_1 is not reducible to any orbit equivalence relation.

Theorem [Kulikov]

$E_1 \leq_B \underline{\bowtie}_c^{<\omega}$.

As corollaries, we have

- $\underline{\bowtie}_c^{<\omega}$ is not reducible to any orbit equivalence relation;
- $\cong_{LO} <_B \underline{\bowtie}_c^{<\omega}$;
- $\underline{\bowtie}_c^{<\omega}$ does not reduce to $\underline{\bowtie}_{\mathrm{LO}}$.

Denote by $\approx_{\mathcal{K}}^{<\omega}$ its associated (analytic) equivalence relation and call it the **(finite) piecewise mutual component relation**.

Theorem [I. - Marcone - Motto Ros]

- $(\trianglelefteq_c^{<\omega})_{CO} \leq_B \precsim_{\mathcal{K}}^{<\omega}$. Then, we have $(\bowtie_c^{<\omega})_{CO} \leq_B \approx_{\mathcal{K}}^{<\omega}$.
- $\cong_{CO} \sim_B \cong_{LO} <_B \approx_{\mathcal{K}}^{<\omega}$.

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Thank you for your attention!